

# Modern Risk Portfolio Optimization: Assets Allocation in Stock using Single Index Model under five Constraints

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**Abstract:** Modern investors have a large range of investment opportunities and investment choices and among them, stocks are very important investment securities. Though having these changes, a lot of investors don't grasp enough investment tools and knowledge to enable them to make reasonable decisions. They are facing a daunting challenge when attempting to determine how to efficiently optimize their portfolios. In this paper, I use the Single Index model which is developed by excel, to analyze 10 popular stocks in America and find the optimal portfolios and minimal risk portfolios under five different constraints. These five constraints are developed to simulate distinct situations in reality in order to make it closer to reality and make our research result more empirical. The article attempts to present a practical solution to the strategic asset allocation problem that investors face and try our best to help investors to make the most sensible choice attaining return and eliminating risk under different situations.

## 1. Introduction

The investment portfolio is a very important topic in finance. A very large number of academic papers have discussed this issue from different aspects. Douglas J. Cumming has discussed the determinants of venture capital portfolio size in banks using empirical evidence [1]. While many scholars chose to research the relationship between individual investor risk aversion and investment portfolio composition [2]. And use risk aversion to evaluate different assets' performance [3]. William Nelson Goetzmann talked about the single-family home in the investment in housing by measuring risk and return [4]. Also, Jeff Grover and Angeline M. Lavin present a practical solution to the strategic asset allocation problem that investors face when attempting to construct an optimal portfolio from a given set of available mutual funds [5]. International equity portfolio investment flows is discussed, as well, for example, based on differences in informational endowments between foreign and domestic investors.[6]

In this paper, facing that many individuals don't have useful tools and knowledge when choosing an investment portfolio, we use risk and return measured by Single Index model, developed by excel, to discuss individual risk investment portfolio composed of 10 popular stocks under different constraints. In order to obtain the optimal portfolio, we use the sharpe ratio to find the maximum return portfolio with the same risk or the minimum risk portfolio with the same return. Finally, we got the minimal risk portfolio the optimal portfolio and the weights of each stock in each portfolio.

The remainder of the article is organized as follows: Section 2 describes data; Section 3 introduces the Single Index Model, along with the constraints; Section 4 performs data analysis; The last section presents our conclusions.

## 2. Data

Twenty years of historical returns are used in this research ranging from May 11th 2001 to May 11th 2021. And we select ten stocks that belong to three parts, technology, energy, consumer defensive,

and consumer cyclical. Also, one (S&P 500) equity index (a total of eleven risky assets) and a proxy for risk-free rate (1-month Fed Funds rate) are used.

In the ten stocks, the technology part involves Qualcomm Incorporated, Akamai Technology, Inc, Oracle Corporation, and Microsoft Corporation. Qualcomm Incorporated engaged in the development and commercialization of foundational technology for the wireless industry worldwide. Akamai Technology provides cloud services for securing, delivering, and optimizing content and business applications over the internet in the United States and internationally. Oracle Corporation provides products and services that address enterprise information technology environments. Microsoft Corporation develops, licenses, and supports software, services, devices, and solutions.

Chevron Corporation, Exxon Mobil Corporation, and Imperial Oil Limited comprise the energy part. Chevron Corporation, through its subsidiaries, engages in integrated energy, chemicals, and petroleum operations worldwide. Exxon Mobil Corporation explores for and produces crude oil and natural gas in the United States and internationally. Imperial Oil Limited explores for, produces, and sells crude oil and natural gas in Canada. It operates through three segments: Upstream, Downstream, and Chemical.

Three companies belong to the Beverage and Food Manufacturing Industry, Coca-Cola Company and PepsiCo, Inc in the Consumer Defensive Sector, McDonald's Corporation in the Consumer Cyclical sector. The Coca-Cola Company, a beverage company, manufactures, markets, and sells various non-alcoholic beverages worldwide. It operates through a network of independent bottling partners, distributors, wholesalers, and retailers, as well as through bottling and distribution operators. PepsiCo, Inc. operates as a food and beverage company worldwide. The company operates through seven segments and distributor networks, as well as directly to consumers through e-commerce platforms and retailers. McDonald's Corporation operates and franchises McDonald's restaurants in the United States and internationally. Its restaurants offer various food products and beverages, as well as a breakfast menu. As of December 31, 2020, the company operated 39,198 restaurants.

All data are downloaded from Yahoo! Finance. I first convert the daily price to daily return. In order to reduce the non-Gaussian effects, I aggregate the daily data to the monthly observations [7]. Based on these observations, I calculate the correlation between these stocks and each stock's annual average return, annual standard deviation, beta, alpha, and residual standard deviation. We can see, apart from the SPX and the two energy corporations, CVX (Chevron Corporation) and XOM (Exxon Mobil Corporation), all the companies have relatively low correlation which will definitely help spread the risk. Annual average return ranges from 5.365%(XOM) to 28.136%(AKAM) and standard deviation ranges from 15.074%(PEP) to 33.283%(QCOM).

### **3. Method**

#### **3.1 Single Index model**

I use the Single Index model to calculate the optimization inputs. An Index Model is a Statistical model of security returns as opposed to an economic, equilibrium-based model. It is widely used and it usually focuses on the estimation of the index coefficients [8]. Single Index Model (SIM) specifies two sources of uncertainty for a security's return: First, systematic (macroeconomic) uncertainty which is assumed to be well represented by a single index of stock returns. Second, unique (microeconomic) uncertainty which is represented by a security-specific random component. Following these principles, it is a very convenient tool to analyze various financial fields such as farming [9]. In this article, we use it to create the most return-to-risk efficient portfolio by analyzing various portfolio combinations based on expected returns (mean) and risk (standard deviations) of the assets. And the portfolio with the largest Sharpe ratio is the optimal one.[10] Essentially, the Index model's theory mitigates a portfolio's overall risk by offsetting the risks of certain stocks with those of other stocks. Compare with the Full Markowitz model's way to calculate the offset the risk, the Index model largely simplify the variance calculating method declining the number needed from  $n$  firm-specific variances and  $n(n-1)/2$  covariances to  $n$  estimates of the firm-specific variances,  $n$  estimates of the sensitivity coefficients

$\beta_i$  and 1 estimate of the variance of the common macroeconomic factor. For the Single Index model, formulas used to calculate the two-stock portfolio expected return:

$$E(r_p) = \alpha_p + \beta_p \times R_m + e_p \quad (1)$$

$$\beta_p = \frac{1}{n} \sum_{i=1}^n \beta_i; \quad (2)$$

$$\alpha_p = \frac{1}{n} \sum_{i=1}^n \alpha_i \quad (3)$$

$$e_p = \frac{1}{n} \sum_{i=1}^n e_i \quad (4)$$

Variance formulas is:

$$\sigma_p^2 = \beta_p^2 \times \sigma_m^2 + \sigma^2(e_p) \quad (5)$$

$$\sigma^2(e_p) = \frac{1}{n} \times \sigma^2(e) \quad (6)$$

While for a ten-stock portfolio, I use matrix and vector to calculate these variables. the set of instruments' average returns is

$$\vec{\mu} = \{\mu_1, \mu_2, \dots, \mu_n\}^T \quad (7)$$

The unknown set of instruments' weights is:

$$\vec{w} = \{w_1, w_2, \dots, w_n\}^T \quad (8)$$

The set of instruments' standard deviations is:

$$\vec{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_n\}^T \quad (9)$$

The set of instruments' betas is:

$$\vec{\beta} = \{\beta_1, \beta_2, \dots, \beta_n\}^T \quad (10)$$

The set of the residuals' standard deviations is:

$$\{\sigma(\varepsilon_1), \sigma(\varepsilon_2), \dots, \sigma(\varepsilon_n)\}^T \quad (11)$$

An auxiliary vector is:

$$\vec{v} = \{w_1\sigma_1, w_2\sigma_2, \dots, w_n\sigma_n\}^T \quad (12)$$

the matrix of instruments' cross-correlation coefficients is:

$$P = \begin{Bmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2n} \\ \dots & \dots & \dots & \dots \\ \rho_{n1} & \rho_{n2} & \dots & \rho_{nm} \end{Bmatrix} \quad (13)$$

So the formula for the full Markowitz model portfolio return is

$$r_p = \vec{w} \cdot \vec{\mu}^T. \quad (14)$$

The formula for the full Markowitz model portfolio standard deviation is

$$\sigma_p = \sqrt{(\sigma_M \beta_p)^2 + \sum_{i=1}^n w_i^2 \sigma^2(\varepsilon_i)}, \quad (15)$$

After getting these two variables, we use them to calculate the sharp ratio in this way

$$\text{Sharpe ratio} = \frac{E(rp) - rf}{\sigma} \quad (16)$$

This is the most return-to-risk efficient portfolio. We further add risk-free Treasury bills to the portfolio and we use the data of the latest month's risk-free return which is very close to zero. I connect these two points into a straight line and that is the capital market line (CML). After that, the client can add their own utility function to find a proper investment portfolio.

### 3.2 Constraint

I also use the full Markowitz model to explore the best investment portfolio under different constraints to make our analysis close to reality. I calculate the data and draw the line of the efficient frontier, minimal risk portfolio, optimal portfolio, and minimal return portfolios frontier for the following five cases of the additional constraints.

The first one is designed to simulate the Regulation T by FINRA, which allows broker-dealers to allow their customers to have positions, 50% or more of which are funded by the customer's account equity:

$$\sum_{i=1}^n |w_i| \leq 2. \quad (17)$$

The second constraint is designed to simulate some arbitrary "box" constraints on weights, which may be provided by the client:

$$|w_i| \leq 1, \text{ for } \forall i. \quad (18)$$

The third one is a "free" problem, without any additional optimization constraints, to illustrate how the area of permissible portfolios in general and the efficient frontier, in particular, look like if you have no constraints.

The fourth additional optimization constraint is designed to simulate the typical limitations existing in the U.S. mutual fund industry: a U.S. open-ended mutual fund is not allowed to have any short positions, for details see the Investment Company Act of 1940, Section 12(a)(3):

$$w_i \geq 0, \text{ for } \forall i. \quad (19)$$

Lastly, we would like to see if the inclusion of the broad index into our portfolio has a positive or negative effect, that we would like to consider an additional optimization constraint:

$$w_i = 0. \quad (20)$$

## 4. Result analysis

After the calculation, I finally got the optimal portfolio and minimal risk portfolio (Table 1) under five different constraints. We can obtain the minimum variance of 11.364% under the constant 1,2,3 and the maximum Sharpe ratio 0.953 under the constraints 2,3. The third constraint is a free one so it is understandable that we can get the most wanted result from it.

I have also used the data to draw the efficient frontiers, capital allocation line (CAL), and the two portfolios (minimal risk portfolio and optimal portfolio) under each constraint. This allows us to intuitively see their different performances under different situations. From this graph we can conclude that the more strict the constraint is, the smaller range in which people can adapt the weight of each risk security and the less likely it that they obtain the minimum risk portfolio and maximum Sharpe ratio portfolio.

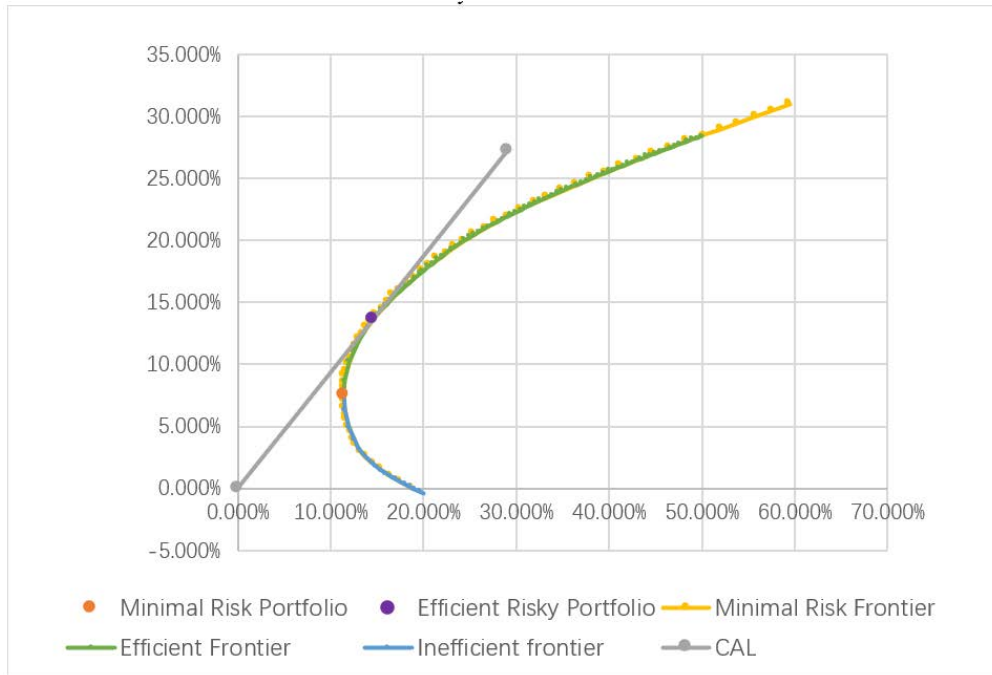


Figure 1. Frontiers and lines under constraint 1

### 5. Conclusion

In this paper, we mainly use a single index model to analyze the best way to allocate capital, especially among risk securities under different constraints for different situations and fields. We can see that the more free the constraint is, the more ideal allocation we can get. However, the promulgation of strict decrees is necessary for some industries to prevent the major financial crisis from happening. And although there is a constraint, we can still get the same minimal risk portfolio under constraints 1, 2 and the same optimal portfolio under constraint 2 as constraint 3.

Although, we successfully provide an example about how to build an optimal portfolio under different constraints to investors. There are still some points we have overlooked. First, it is inaccurate to use only one parameter to predict the stocks’ return in the future. We need to add more to make the result more precise.

Table 1. Short cut keys for the template

IM (Constr1 ):	SPX	WFC	LUV	PGR	LST R	CSC O	TD CN	PG	MSF T	KO	MCD
MinVar	0.129 117	- 0.042 15	- 0.025 02	- 0.005 24	- 0.000 86	0.033 001	0.090 97	- 0.013 79	0.297 569	0.367 243	0.169 162
MaxSha rpe	- 0.484 33	0.043 932	0.062 009	0.065 315	0.217 388	0.044 41	- 0.015 62	0.038 136	0.204 765	0.335 287	0.488 712
IM (Constr2 ):	SPX	WFC	LUV	PGR	LST R	CSC O	TD CN	PG	MSF T	KO	MCD
MinVar	0.129 117	- 0.042 15	- 0.025 02	- 0.005 24	- 0.000 86	0.033 001	0.090 97	- 0.013 79	0.297 569	0.367 243	0.169 162

MaxSharpe	-	0.081	0.079	0.107	0.297	0.101	-	0.072	0.254	0.403	0.577
(Constr3):	0.93839	83	292	313	742	571	0.03676	451	281	052	619
IM											
MinVar	0.129117	0.04215	0.02502	0.00524	0.00086	0.033001	0.09097	0.01379	0.297569	0.367243	0.169162
MaxSharpe	-	0.081	0.079	0.107	0.297	0.101	-	0.072	0.254	0.403	0.577
(Constr4):	0.93839	83	292	313	742	571	0.03676	451	281	052	619
IM											
MinVar	0.014453	0	0	0	0	0.033952	0.093591	0	0.306143	0.377824	0.174036
MaxSharpe	0	0.006496	0.052612	0.019718	0.143448	0	0	0	0.121793	0.23708	0.418854
(Constr5):											
IM											
MinVar	0	0.03353	0.02251	0.004906	0.015993	0.049203	0.106177	0.00529	0.313674	0.38657	0.184813
MaxSharpe	0	0.017211	0.057892	0.032531	0.169239	0.01372	0.13775	0.010106	0.14387	0.26725	0.453374

Return	StDev	Sharpe
7.543%	11.364%	0.664
13.634%	14.546%	0.937
Return	StDev	Sharpe
7.54%	11.364%	0.664
15.56%	16.322%	0.953
Return	StDev	Sharpe
7.543%	11.364%	0.664
15.558%	16.322%	0.953
Return	StDev	Sharpe
8.385%	11.527%	0.727
12.03%	13.80%	0.872
Return	StDev	Sharpe
7.880%	11.395%	0.692
12.911%	14.587%	0.885

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